The Richard Stockton College of New Jersey
Mathematical Mayhem 2013
Group Round

March 23, 2013

Name: ____

Name: ____

Name: ____

High School: ____

Instructions:

- This round consists of **5** problems worth **16** points each for a total of **80** points.
- Each of the 5 problems is free response.
- Write your complete solution in the space provided including all supporting work.
- No calculators are permitted.
- This round is **75 minutes** long. **Good Luck!**

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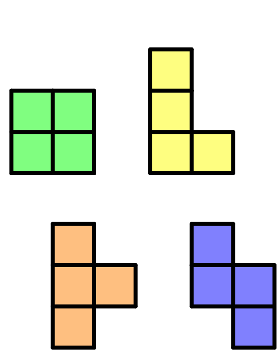
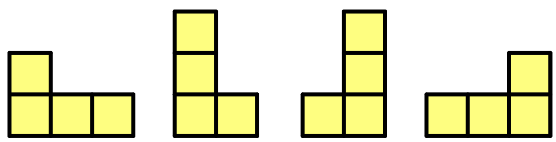
Problem #	1	2	3	4	5	Total
Points Earned						

♠ Group Round ♠

Problem 1. A polyomino is a contiguous shape formed by gluing together squares edge to edge. A polyomino made up of 4 squares is called a tetromino. There are 5 different tetrominoes, as shown below.

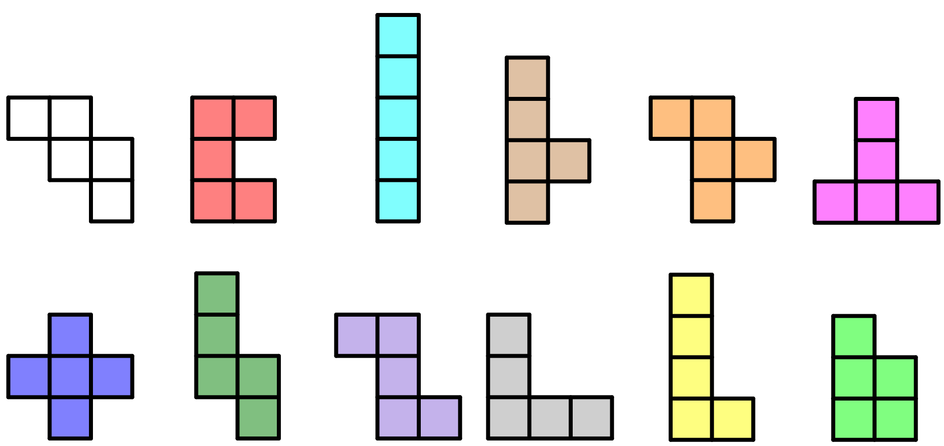


Flipping or rotating a tetromino does not make it a different tetromino. For instance, the four tetrominoes shown below are all considered to be the same tetromino.

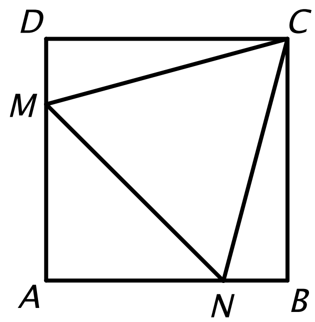


A polynomino made up of 5 squares is called a pentomino. How many different pentominoes are there?

Solution to Question 1. There are $\boxed{12}$ pentominoes, pictured below.



Problem 2. In the figure below $ABCD$ is a square and CMN is an equilateral triangle. If the area of $ABCD$ is one square inch, what is the area of CMN in square inches?



Solution to Question 2. Let $DM = NB = x$. Then $AM = AN = 1 - x$.

$$\begin{aligned}\text{Area of } \triangle CMN &= \text{Area of } \square ABCD - \text{ Area of } \triangle ANM - \text{ Area of } \triangle NBC - \text{ Area of } \triangle CDM \\ &= 1 - \frac{1}{2}(1-x)^2 - \frac{x}{2} - \frac{x}{2} \\ &= \frac{1}{2}(1-x^2).\end{aligned}$$

Let y be the length of each side of the equilateral triangle CMN. By the Pythagorean theorem, we have

$$\begin{aligned} x^2 + 1^2 &= y^2, \\ (1-x)^2 + (1-x)^2 &= y^2. \end{aligned}$$

From these two equations, we get

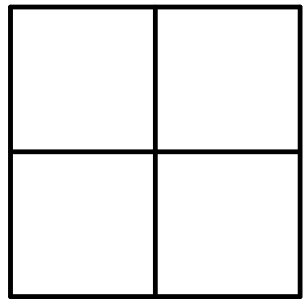
$$2(1-x)^2 = x^2 + 1,$$

which can be simplified to

$$x^2 - 4x + 1 = 0,$$

with two solutions, $2 - \sqrt{3}$, and $2 + \sqrt{3}$. Since $2 + \sqrt{3} > 1$, we have to choose $2 - \sqrt{3}$. Then the area of $\triangle CMN$ is $\boxed{2\sqrt{3} - 3}$.

Problem 3. How many total squares are there in a 100×100 grid? How many total squares are there in a $n \times n$ grid? For example, there are 5 squares in the 2×2 grid shown below.



Solution to Question 3. The number of squares in an $n \times n$ grid is the sum

$$1^2 + 2^2 + \ldots + (n-1)^2 + n^2 = \boxed{\frac{n(n+1)(2n+1)}{6}}.$$

For $n = 100$, this is $\boxed{338350}$.

Problem 4.

(A.) When I sum five numbers in every possible pair combination, I get the values:

$$0, 1, 2, 4, 7, 8, 9, 10, 11, 12.$$

What are the original 5 numbers?

(B.) When I sum a different set of five numbers in every possible group of 3, I get the values:

$$0, 3, 4, 8, 9, 10, 11, 12, 14, 19.$$

What are the original 5 numbers?

(C.) Is it possible to find a set of 5 numbers that when summed in every possible pair combination results in the sums

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10?$$

Is it possible to find a set of 5 numbers that when summed in every possible group of 3 results in those sums? For each situation, find an example or prove it's impossible.

Solution to Question 4.

(A.) The sum of the pairwise sums is 64, and this counts each of the original five numbers four times, so the sum of the original five numbers is 16. The sum of the largest two of the original five numbers is 12, and the sum of the smallest two is 0, so the middle number is 4. (i.e. $16 - 12 - 0 = 4$) The sum of the largest and middle is the second largest sum, 11, so the largest must be 7, and the second largest is 5. (i.e. $12 - 7 = 5$) In order for the second-smallest sum to be 1, one of the numbers has to be -3, so the numbers are $\boxed{-3, 3, 4, 5, 7}$.

(B.) The sum of the triples is 90, and this counts each of the original five numbers six times, so the sum of the original five numbers is 15. The largest triple sum is 19, so the two not included sum to -4. The next-largest triple is 14, so the pair not included sum to 1, etc. So the pairwise sums are -4, 1, 3, 4, 5, 6, 7, 11, 12, 15. The sum of the largest two of the original five is 15, and the sum of the smallest two is -4, so the middle number is 4. The sum of the largest and middle is 12, so the largest must be 8, and the second largest must be 7. The sum of the smallest and middle is 1, so the smallest must be -3, so the numbers are $\boxed{-3, -1, 4, 7, 8}$.

(C.) $\boxed{\text{No}}$, because the sum of 1-10 must be four times the sum of the five numbers, but the sum of 1-10 is 55, which is odd, and so it is not a multiple of four (as would be required for sums of pairs) or a multiple of six (as would be required for sums of triples). Moreover, any sequence of ten consecutive integers sums to an odd number, so no such sequence is possible.

Problem 5.

(A.) There are nine balls, identical in shape and identical in weight, except one, which is slightly heavier than the others. You have a balance scale. Identify the heavier ball using only two weighs. That is, you may compare the weights of two sets of balls, and then you may compare the weights of two other sets of balls.

(B.) Suppose now that there are b balls which are identical in shape and weight except one. If you are allowed 3 weighs, what is the largest number b for which you would be able to identify the heavier ball? How would you do it?

(C.) If you are allowed n weighs, what is the largest number b for which you would be able to identify the heavier ball?

Solution to Question 5. For the general procedure, consider the case with 3^n balls. Partition the balls into 3 sets, each with 3^{n-1} balls and compare two of those sets on the with the scale. If the scale tells you that the sets have the same weight, then the heavier ball is in the set of 3^{n-1} balls that you did not put on the scale. If the scale tells you that one of the sets weighs more than the other, then the heavier ball is in the heavier set of 3^{n-1} balls. In both cases, a single weigh reduces the number of candidates for the heavier ball by $1/3$rd. Using this procedure, it is possible to narrow a set of 9 balls down to a single ball in 2 weighs, a set of 27 balls to 1 ball in 3 weighs, and a set of 3^n balls to 1 ball in n weighs.