Stockton University
Mathematical Mayhem 2018
Individual Round with Answers

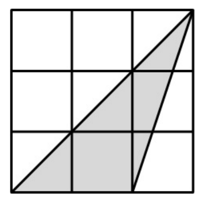
April 14, 2018

Instructions:

- This round consists of **18** problems worth a total of **80** points, made up of 8 Appetizers worth 3 points each, 7 Entrées worth 5 points each, and 3 Desserts worth 7 points each.
- Each of the 18 problems is multiple choice, and each problem comes with **5** possible answers.
- For each problem, **mark your answer on the answer sheet**.
- You are not required to show any work this round.
- No calculators are permitted.
- This round is **75 minutes** long. **Good Luck!**

♣ Appetizers ♣

Problem 1. In the diagram, each small square is 1 cm by 1 cm. What is the area of the shaded region, in square centimeters?



(A.) 2.75 (B.) 3 (C.) 3.25 (D.) 4.5 (E.) 6

Solution to Question 1. (B).

Problem 2. Which value of n makes the following identity true $a^n = [\sqrt[3]{\sqrt[6]{a^9}}]^4[\sqrt[6]{\sqrt[3]{a^9}}]^4$?
(A.) 2 (B.) 16 (C.) 8 (D.) 12 (E.) 4

Solution to Question 2. (E). A way to simplify is to rewrite the radicals by fractional exponents and then to multiply the exponents. We get $(a^{\frac{1}{18}})^{36}(a^{\frac{1}{18}})^{36} = a^4$.

Problem 3. What is the value of $\frac{2018^2-3\cdot 2018-4}{2018+1}$?
(A.) 2014 (B.) 2015 (C.) 2016 (D.) 2017 (E.) 2018

Solution to Question 3. (A).

Problem 4. A gold bar is a rectangular solid measuring $2 \times 3 \times 4$. It is melted down and three equal cubes are constructed from this gold. What is the length of the side of each cube?
(A.) 1 (B.) $\sqrt[3]{2}$ (C.) $\sqrt{2}$ (D.) 2 (E.) 4

Solution to Question 4. (D).

Problem 5. How many integers n are there that satisfy the inequality $\frac{1}{5} < \frac{4}{n} < \frac{1}{3}$?
(A.) 9 (B.) 1 (C.) 7 (D.) 6 (E.) 8

Solution to Question 5. (C).

Problem 6. In a group of cows and chickens, the number of legs is 14 more than twice the number of heads. How many cows are there?
(A.) 10 (B.) 5 (C.) 14 (D.) 12 (E.) 7

Solution to Question 6. (E). Let x be the number of cows and y be the number of chickens. $4x + 2y$ is the total number of legs. $4x + 2y = 14 + 2(x + y)$. $2y$ will be canceled out from the equation. We will get $x = 7$. Please notice that the solution of y can not be determined.

Problem 7. Out of all positive integers x and y, what is the smallest value of $x + y$ for which $x + y \leq xy$?
(A.) 8 (B.) 4 (C.) 10 (D.) 2 (E.) 5

Solution to Question 7. (B). When $x = 1$, the equation does not hold since $1 + y > y$. The next smallest possible value for $x + y$ occurs when $x = 2$ and $y = 2$. Indeed $2 + 2 \leq 2(2)$ and so the smallest possible sum is 4.

Problem 8. Let $f(x) = \frac{x}{x+3}$ for $x \neq -3$. How many solutions are there to the equation $f(f(x)) = x$?
(A.) 0 (B.) 1 (C.) 2 (D.) 3 (E.) 4

Solution to Question 8. (C).

$$f(f(x)) = \frac{\frac{x}{x+3}}{\frac{x}{x+3} + 3} = \frac{x}{x + 3(x + 3)} = \frac{x}{4x + 9}.$$

So, $x = f(f(x)) = \frac{x}{4x+9}$ and $x = 4x^2 + 9x$. This gives $0 = 4x(x + 2)$ and so $x = 0$ or $x = -2$. Hence, there are 2 solutions.

◊ Entrées ◊

Problem 9. What is the result after $\frac{a^{-4}-b^{-4}}{a^{-2}-b^{-2}}$ is simplified?
(A.) $a^{-6}-b^{-6}$ (B.) $a^{-2}-b^{-2}$ (C.) a^2-b^2 (D.) a^2+b^2 (E.) $a^{-2}+b^{-2}$

Solution to Question 9. (E).

Problem 10. The length of the rectangle R is 10 percent more than the side of the square S. The width of the rectangle is 10 percent less than the side of the square S. What is the ratio of the area of R to the area of S?
(A.) 99:100 (B.) 101:100 (C.) 1:1 (D.) 199:100 (E.) 201:200

Solution to Question 10. (A). Let x be the length of R, y be the width of R, and s be the side of S. $x = 1.1s$, and $y = 0.9s$. The area of R is $xy = 1.1 \cdot 0.9s^2$. The ratio of the area of R to the area of S is 0.99:1 or 99:100.

Problem 11. The cross-sections of a particular tree are circles whose areas grow as a linear function of time. The diameter of the tree is 2 feet in 1938 and 4 feet in 1998. What is its diameter in feet in 2018?
(A.) $2\sqrt{5}$ (B.) $\sqrt{3}$ (C.) 2 (D.) $\sqrt{5}$ (E.) $2\sqrt{3}$

Solution to Question 11. (A). Let $a(t)$ and $d(t)$ be the area and diameter of the tree as functions of time where t is the number of year after 1938. Then $a(t) = \pi(d(t))^2/4 = mt + b$ and the diamter $d(t) = \sqrt{4mt/\pi + 4b/\pi}$. So $d(0) = 2 = \sqrt{4b/\pi}$ and so $b = \pi$. Further, $d(60) = 4 = \sqrt{240m/\pi + 4}$. Hence $m = \pi/20$ and $d(t) = \sqrt{t/5 + 4}$. That is, $d(80) = \sqrt{80/5 + 4} = \sqrt{20} = 2\sqrt{5}$.

Problem 12. What is the value of

$$\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{5}\right)\left(1+\frac{1}{6}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{8}\right)\left(1+\frac{1}{9}\right)?$$

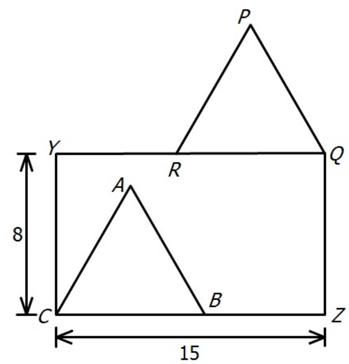
(A.) $\frac{1}{2}$ (B.) 5 (C.) 9 (D.) $9\frac{1}{8}$ (E.) $\frac{10}{9}$

Solution to Question 12. (B). $\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{5}\right)\left(1+\frac{1}{6}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{8}\right)\left(1+\frac{1}{9}\right) = \frac{3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10}{2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9} = \frac{10}{2} = 5$

Problem 13. The number 1234567891011 ... 585960, which consists of the first 60 positive integers written in order to form a single number with 111 digits, is modified by removing 100 of its digits while keeping the order of remaining 11 digits unchanged. What is the smallest number that can be obtained in this way?
(A.) 12345 (B.) 123450 (C.) 123456 (D.) 11111 (E.) 12345678910

Solution to Question 13. (B). We seek to put the smallest possible digits on the left. Notice that we may choose the leftmost 5 digits 0's and the remaining 6 digits must be chosen from 51525354555657585960. The smallest digit that can now be leftmost of the last 6 digits is 1, followed by 2, then 3, then 4, and then 5. The rightmost digit can be 0, and so the smallest possible number is 123450.

Problem 14. In the diagram, $YQZC$ is a rectangle with $YC = 8$ and $CZ = 15$. Equilateral triangles ABC and PQR, each with side length 9, are positioned as shown with R and B on sides YQ and CZ, respectively. What is the length of AP?



(A.) 10 (B.) $\sqrt{117}$ (C.) 9 (D.) 8 (E.) $\sqrt{72}$

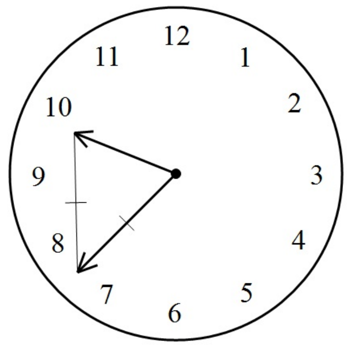
Solution to Question 14. (A). The translation that takes C to A also takes R to P. Since translation preserves distance, $AP = CR$. Since QR is length 9, RY is length 6. By the Pythaogrean Theorem, RC is length 10.

Problem 15. If $(1.0000042376)^2 = 1.00000xyz521795725376$, what is the value of $x + y + z$?
(A.) 15 (B.) 16 (C.) 17 (D.) 18 (E.) 19

Solution to Question 15. (E). Let $t = .0000042376$ and $(1 + t)^2 = 1 + 2t + t^2$. Observe that t^2 is too small to affect the first three nonzero digits of $2t$, so $xyz = 847$ and $x + y + z = 19$.

♡ Desserts ♡

Problem 16. At some time between 9:30 and 10 o'clock the triangle determined by the minute hand and the hour hand is an isosceles triangle (see diagram). If the two equal angles in the triangle are twice as large as the third angle, what is the time?



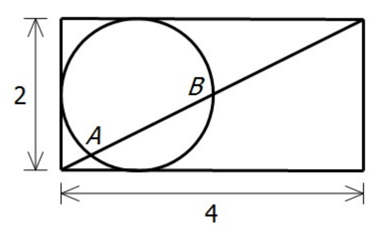
(A.) 9:39 (B.) 9:38 (C.) 9:37 (D.) 9:36 (E.) 9:35

Solution to Question 16. (D). Note that the angle made by the minute and hour hands is 72° which is one fifth of the total 360° around the clock. Let m be the number of minutes passed 9 of the time. Then $45+5m/60$ gives the position of the hour hand in minutes passed 9 and m gives the position of the minute hand. Further, $72^\circ = 360^\circ/5$ corresponds to 60 minutes$/5$ = 12 minutes. Hence

$$45 + 5m/60 = m + 12.$$

Solving for m gives $m = 33(60/55) = 36$ and the current time is 9:36.

Problem 17. A circle is tangent to three sides of a rectangle having side lengths 2 and 4 as shown. A diagonal of the rectangle intersects the circle at points A and B. What is the length of the segment AB?



(A.) $\sqrt{5}$ (B.) $\sqrt{5}-\frac{1}{6}$ (C.) $\sqrt{5}-\frac{1}{5}$ (D.) $\frac{4\sqrt{5}}{5}$ (E.) $\frac{5\sqrt{5}}{6}$

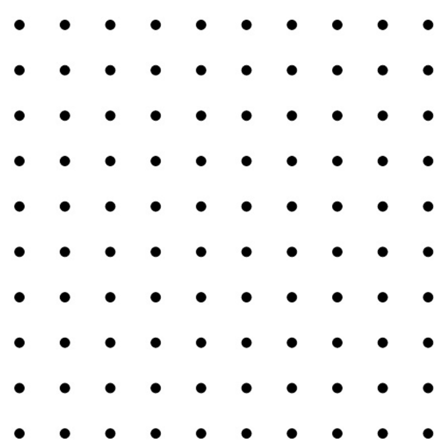
Solution to Question 17. (D). Let O be the center of the circle, C be the point of tangency between the circle and the left side of the rectangle, D be the point of tangency between the circle and the bottom of the rectangle, and E be the bottom left corner of the rectangle. Then $OCED$ is a square since the lengths of OC and OD are equal and the interior angles at C, E, and D are all right angles. So, if we can setablish a coordinate system by setting O to be the origin. Further, we can let C be $(-1,0)$, E be $(-1,-1)$, and D be $(0,-1)$. Then the upper right corner of the rectangle is $(3,1)$. Hence, the line segment is the line through $(-1,-1)$ and $(3,1)$. This is given by the equation $y = \frac{x}{2} - \frac{1}{2} = \frac{1}{2}(x-1)$. Further, the circle is given by the equation $x^2 + y^2 = 1$. We solve for the points of intersection A and B. Note that $y^2 = 1 - x^2 = \frac{1}{4}(x-1)^2$. Further,

$$\begin{aligned} 1 - x^2 &= \frac{1}{4}(x-1)^2 = \frac{1}{4}(x^2 - 2x + 1) = \frac{1}{4}x^2 - \frac{1}{2}x + \frac{1}{4} \\ 0 &= \frac{5}{4}x^2 - \frac{1}{2}x - \frac{3}{4} \\ 0 &= 5x^2 - 2x - 3 \end{aligned}$$

Note that $x = \frac{2 \pm \sqrt{4 - 4(-3)(5)}}{10} = \frac{2 \pm 8}{10}$ and so $x = -3/5$ or $x = 1$. Solving for the corresponding y values, we get $-4/5 = \frac{1}{2}((-3/5) - 1)$ and $0 = \frac{1}{2}(1 - 1)$. Hence the points of intersection are $(-3/5, -4/5)$ and $(1, 0)$, so we can solve for the length AB by using the distance formula:

$$AB = \sqrt{(-3/5 - 1)^2 + (-4/5 - 0)^2} = \sqrt{64/25 + 16/25} = \frac{4\sqrt{5}}{5}.$$

Problem 18. In the square array of dots with 10 rows and 10 columns of dots shown below each dot is colored either red or blue. When two dots of the same color are adjacent in the same row or column, they are joined by a line segment that is the same color as the dots. If they are adjacent, but are of different colors, they are joined by a green line segment. In total, there are 52 red dots. There are 2 red dots at corners and an additional 16 red dots that are on the edges (but not corners) of the array. The rest of the red dots are inside the array. There are 98 green segments. How many blue line segments are there?



(A.) 36 (B.) 37 (C.) 38 (D.) 39 (E.) 40

Solution to Question 18. (B). There are 9 line segments in each row and in each column, so there are 180 line segments total. Let B be the number of blue segments and R be the number of red segments. Then $B + R + 98 = 180$, and so $B + R = 82$, as there are 98 green line segments.

Coming out of a red dots, there can only be a green line segment or a red line segment. We count the total number of line segments starting from red dots. Note that in this total, the green segments are counted once and the red segments are counted twice, as the red segments have both ends at red dots. There are two edges coming from a corner dots, three edges from a non-corner edge dot, and four edges coming from an interior dot.

So, the total number of edges coming from red dots, with red edges counted twice and green edges counted once, is:

$$2(2) + 3(16) + 4(52 - 16 - 2) = 2R + 98.$$

Hence, $4 + 48 + 136 = 188 = 2R + 98$, and so $R = 45$. Finally, $B = 82 - 45 = 37$.